DIFFERENTIALS AND INTEGRATION

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ABSTRACT. In this short set of notes, we detail the relationship between differentials and integration for the purposes of single variable calculus, without going into the more general theories of measures or differential forms. The primary goal is to provide curious A-level students with a rigorous formulation which justifies the methods behind the techniques they use for integration.

1. DIFFERENTIALS

Recall that for a function f, the differential df is a function of two independent variables x and dx, defined by

$$df(x, dx) = f'(x)\, dx. \tag{1}$$

In contrast to df, the variable dx here has no underlying meaning (for now), it is just an independent variable, and we could equivalently write

$$df(x, h) = f'(x)\, h.$$

Notice that using dx rather than h, we nicely get that $\frac{df}{dx} = f'(x)$. (For this to make sense, the derivative $f'(x)$ is defined separately first, say, as the unique real number such that

$$f(x + dx) = f(x) + f'(x)\, dx + o(dx),$$

or as the limit of $(f(x + dx) - f(x))/dx$ as $dx \to 0$. What's important here is that $\frac{df}{dx}$ is not "just another notation" for the derivative, but df and dx have separate meanings unto themselves.)

Remark 1.1. The intuition behind the differential is to calculate the approximate change of a function near a specific value, by approximating the function with a line there. In other words, when dx is "small", we have

$$f(x + dx) \approx \underbrace{f(x)}_{\text{old value}} + \underbrace{df(x, dx)}_{\text{approx change}} = f(x) + f'(x)\, dx,$$

which is a straight line if we treat x as constant and dx as the variable. E.g., with $f(x) = x^2$, we have

$$(x + dx)^2 \approx x^2 + 2x\, dx,$$

so when $x = 2$ (say), we have

$$(2 + dx)^2 \approx 4 + 4\, dx.$$

Indeed, $2.01^2 = 4.0401$, and our approximation gives us $4 + 4(0.01) = 4.04$ (refer to figure 1). The derivative is the slope (or gradient) of our approximation line.

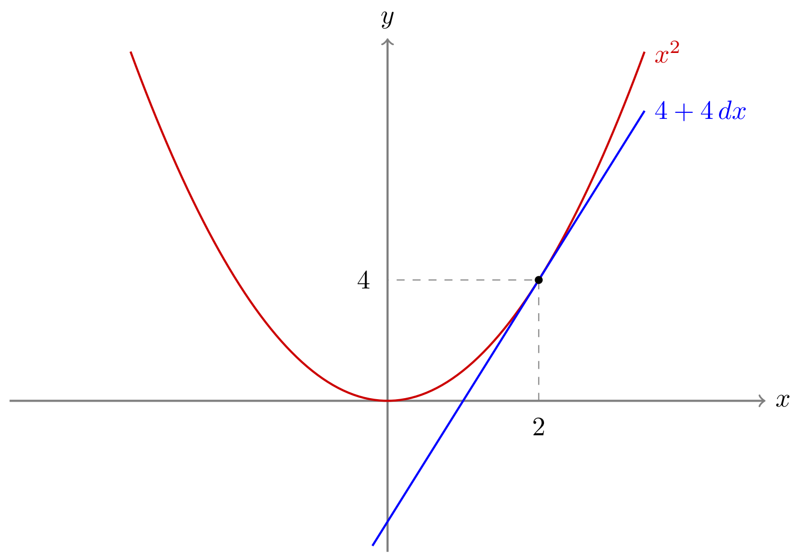

FIGURE 1. Plot of x^2 and $4 + 4\,dx$ on the same axes, notice that for points close to $x = 2$, they are very close.

We will abuse notation slightly and write $d(f(x))$ instead of $df(x, dx)$, where it is understood that the second variable is always denoted by prepending a d to the independent variable. This is so that we may write things like

$$d(\sin^2 x) = 2\sin x \cos x\,dx \qquad \text{instead of} \qquad d(\sin^2)(x, dx) = 2\sin x \cos x\,dx.$$

The use of dx as an independent variable here is indicative of the fact that, if we want, we may think of x as a function of some other variable, say t. In that case, we would also have

$$d(x(t)) = x'(t)\,dt, \tag{2}$$

and substituting this for the independent variable dx in (1), we obtain

$$df\big(x, d(x(t))\big) = f'(x(t))\,d(x(t)) = f'(x(t))\,x'(t)\,dt.$$

The *chain rule* guarantees that this is the same as the differential of $f(x(t))$, i.e., the chain rule is the statement that

$$d(f(x(t))) = df\big(x(t), d(x(t))\big)$$

or without notational abuse, $d(f \circ x)(t, dt) = df\big(x(t), dx(t, dt)\big)$.

Example 1.2. If $f(x) = \sin(x)$ and $x(t) = t^2 + 1$, then

$$d(f(x)) = \cos x\,dx \qquad \text{and} \qquad d(x(t)) = 2t\,dt,$$

thus

$$\begin{aligned} d(f(x(t))) &= d(\sin(t^2 + 1)) \\ &= \cos(t^2 + 1) \cdot 2t\,dt \\ &= f'(x(t))\,d(x(t))). \end{aligned}$$

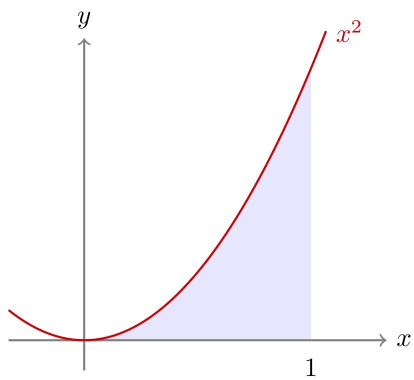

FIGURE 2. The area represented by $\int_0^1 x^2\,dx$.

In summary, we have that if x is an independent variable, then

$$d(f(x)) = f'(x)\,dx$$

where dx is also an independent variable. But if instead, x depends on t, then the same statement is true if we reinterpret x as $x(t)$, and dx as the differential of $x(t)$ (rather than an independent variable):

$$d(f(x(t))) = f'(x(t))\,d(x(t)).$$

Therefore, we could say that

$$d(f(x)) = f'(x)\,dx$$

is in a sense, always true, whether x, dx are independent variables or whether x is a function of some other variable and dx denotes its differential.

2. Integration

Integral calculus has a completely different goal in mind to differential calculus: that of finding the area under a curve. We denote the area bounded by the x-axis, the curve $y = f(x)$ and the lines $x = a$ and $x = b$ by

$$\int_a^b f(x)\,dx.$$

Notice $f(x)\,dx$ looks suspiciously like a differential—this is not a coincidence—more on that later. The usual way we define the area formally is using a Riemann–Darboux sum, which essentially involves an approximation of the area by rectangles from above and below, and letting the number of rectangles become infinitely large. For instance, to find $\int_0^1 x^2\,dx$, the area under x^2 between 0 and 1 (figure 2), we first find the *upper-rectangle sum*

$$U(n) = \sum_{k=1}^{n} \frac{1}{n}\cdot f\left(\frac{k}{n}\right) = \frac{1}{n}\sum_{k=1}^{n}\frac{k^2}{n^2} = \frac{1}{n^3}\left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right) = \frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2},$$

depicted in figure 3. This will be an over-estimate to the "true" value of the desired area $A = \int_0^1 x^2\,dx$, using n rectangles.

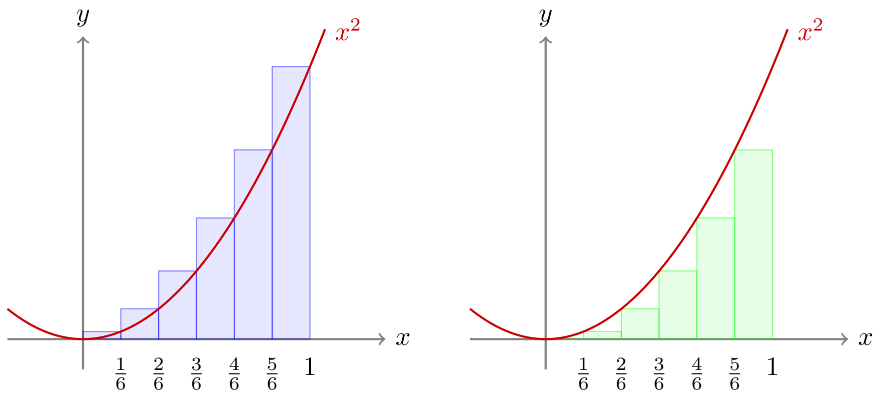

FIGURE 3. Illustration of the upper-rectangle and lower-rectangle sums with $n = 6$.

Analogously, we determine the *lower-rectangle sum*

$$L(n) = \sum_{k=1}^{n} \frac{1}{n} \cdot f\left(\frac{k-1}{n}\right) = \frac{1}{n} \sum_{k=1}^{n} \frac{(k-1)^2}{n^2} = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2},$$

and this will be an under-estimate to A, no matter the value of n. Thus for all n, we have

$$L(n) \leqslant A \leqslant U(n),$$

i.e.,

$$\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2} \leqslant A \leqslant \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}.$$

As n becomes large, we can imagine the rectangles in figure 3 becoming thinner and thinner, so that both the upper-rectangle and lower-rectangle sums get closer and closer to the desired area. If we think about the expressions above, as n becomes large, it is clear that the varying terms all become negligible, so the only sensible value we can assign to A is that of $\frac{1}{3}$. In other words, we have

$$\int_0^1 x^2 \, dx = \frac{1}{3},$$

and we've integrated x^2! This reasoning can be generalised to encompass a large class of functions, the so-called (Riemann) *integrable* functions. To be integrable, a function needn't be differentiable (i.e., it doesn't need to have a derivative, or look "smooth"), it doesn't even have to be continuous (i.e., it can have jumps). It just needs the upper-rectangle and lower-rectangle sums to approach the same number.

So what does differentiation have to do with this? Enter:

3. The Fundamental Theorem of Calculus

As we've seen, the process of finding integrals (areas under curves) from first principles is quite tedious. Simply integrating x^2 relied on the fact that we can evaluate sums like $\sum k^2$. How can we hope to find something like $\int_1^2 \log x \, dx$ or $\int_{-\pi}^{\pi} x \cos x \, dx$? We'd have to evaluate some rather complicated sums.

Miraculously, we can bypass all this with the help of differentiation. The *fundamental theorem of calculus* (FTC) links the two ideas together. The gist of the

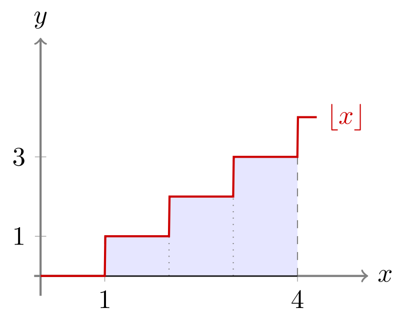

FIGURE 4. $\int_1^4 \lfloor x \rfloor \, dx$

fundamental theorem is this: in order to find the integral $\int_a^b f(x)\,dx$, if we can find *another function* F such that $F' = f$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.[1]

Remark 3.1. Indeed, in the case of our x^2 example, we could have found our desired area by simply noting that $F(x) = \frac{1}{3}x^3$ does the job (i.e., $F'(x) = x^2$). Indeed,

$$\int_0^1 x^2\,dx = F(1) - F(0) = \frac{1}{3} - 0 = \frac{1}{3}.$$

We call F a *primitive* of f.

Remark 3.2. The fundamental theorem tends to make people think of integration as the process of "reversing differentiation", i.e., obtaining primitives. While in most situations this is the easiest way to evaluate integrals, it isn't always necessarily the case. For one thing, primitives cannot always be found. Also, certain functions can be integrated easily using other means, and the fundamental theorem needn't be involved. For instance, the *integer part* $\lfloor x \rfloor$ gives the largest integer smaller than x (e.g., $\lfloor 3.6 \rfloor = 3$). This is easily integrated, it's basically made up of rectangles already (see figure 4). We have $\int_1^4 \lfloor x \rfloor \, dx = 1 \times 1 + 1 \times 2 + 1 \times 3 = 6$.

Here is the more precise statement of the fundamental theorem. Usually the first part is referred to as the FTC1, and the second part (which justifies our technique) as the FTC2.

Theorem 3.3 (Fundamental theorem of calculus)**.** *Let f be a real-valued function defined for all $x \in [a, b]$. Then*

(i) If f is continuous, the function $F(x) = \int_a^x f(t)\,dt$ is a primitive of f, and
(ii) if $F(x)$ is any primitive of $f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Examples 3.4. Since $\frac{d}{dx}(\frac{x^4}{4} + 1) = x^3$, then

$$\int_0^1 x^3\,dx = \left(\frac{1^4}{4} + 1\right) - \left(\frac{0^4}{4} + 1\right) = \frac{1}{4}.$$

Similarly, since $(-\cos)' = \sin$, then

$$\int_0^\pi \sin x\,dx = (-\cos\pi) - (-\cos 0) = 2,$$

i.e., the area under each arc of a sine wave is 2.

[1]We will not prove it here, look at chapters 10 and 11 of these notes for some more intuition.

4. Symbolic Integration

In this section, we discuss techniques for obtaining primitives, i.e., reversing differentiation (this process is called *symbolic* integration). Because it will make the theory more elegant, we will not think of the process as described in the last section, where we found F such that $F' = f$, but rather, as finding F such that $dF = f(x)\,dx$. Of course, this is equivalent, but now the differential operator d is the centre of attention, and rather than being given a function f, we are given (what we assume is) a differential df, and we want to "undo" the effect of the operator d, obtaining f. We denote the primitive of df by $\int df$, so that

$$\int df = f,$$

and this way $\int$ and d are opposites. Now, what we have just written here isn't entirely correct, since the differential operator d is not injective, thus it is fallacious to speak of "the" primitive of df (since there are possibly many to choose from). In particular, the differentials of $f(x)$ and $f(x) + c$, where c is any constant, are the same, so $f(x)$ and $f(x) + c$ are both primitives of df. But it turns out that if we account for this constant difference, then primitives are uniquely determined.

Indeed, if f_1 and f_2 are both primitives of df, then $d(f_1 - f_2) = df_1 - df_2 = df - df = 0$, so the differential of $f_1 - f_2$ is zero. It follows (by the mean-value theorem) that $f_1 - f_2$ is a constant function, say equal to c for all x, so that we have $f_1(x) = f_2(x) + c$ for all x.

Thus we write

$$\int df = f + c,$$

where c represents any constant, and this way, all other possible primitives are also incorporated.

Examples 4.1. We can evaluate some primitives $\int f(x)\,dx$ simply by recognising $f(x)\,dx$ as the differential of a known function. Here are a few examples:

$$\int \cos x\,dx = \int d(\sin x) = \sin x + c$$
$$\int x^3\,dx = \int d\Big(\frac{x^4}{4}\Big) = \frac{x^4}{4} + c$$
$$\int \frac{dx}{x} = \int d(\log x) = \log x + c$$
$$\int e^x\,dx = \int d(e^x) = e^x + c$$

Other common primitives are given in the MATSEC booklet. They can be combined with the following useful rules, which we will prove later.

I. (*Logarithmic derivative*)

$$\int \frac{f'(x)}{f(x)}\,dx = \log f(x) + c.$$

II. (*The affine argument rule*) If $\int f(x)\,dx = F(x) + c$, then

$$\int f(ax + b)\,dx = \frac{F(ax + b)}{a} + c.$$

Another important fact about (symbolic) integration is its *linearity*: for any functions f, g and any real number a, we have

$$\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx, \quad \text{and} \quad \int a\,f(x)\,dx = a\int f(x)\,dx,$$

which follows from the linearity of the operator d.

What if we don't recognise $f(x)\,dx$ as the differential of some other function? For instance, what is

$$\int \sin x \cos x\,dx?$$

This is where the substitution theorem comes in handy.

5. Substitution

The substitution theorem is essentially a reformulation of our observation at the end of the first section, on carefully interpreting statements about differentials so that they are always true. In order to state it, we will need the following piece of notation.

Notation. For any X, let

$$\int_{x \leftarrow X} f(x)\,dx$$

denote the expression obtained by replacing x with X after $\int f(x)\,dx$ is evaluated. For example,

$$\int_{x \leftarrow r^2+3} x^2\,dx = \int_{x \leftarrow r^2+3} d\Big(\frac{x^3}{3}\Big) = \frac{(r^2+3)^3}{3} + c.$$

Now we can state the theorem.

Theorem 5.1 (Substitution theorem)**.** *Let $f(x)$ and $u(x)$ be two differentiable functions. Then*

$$\int f'(u(x))\,d(u(x)) = \int_{u \leftarrow u(x)} f'(u)\,du,$$

where $\int_{x \leftarrow X} f(x)\,dx$ denotes the expression obtained by replacing x with X after $\int f(x)\,dx$ is evaluated.

Proof. Recall that

$$\int f'(u)\,du = \int d(f(u)) = f(u) + c,$$

so replacing the independent variable u with the function $u(x)$, we obtain

$$\int_{u \leftarrow u(x)} f'(u)\,du = f(u(x)) + c.$$

But also from section 1, we saw that $d(f(u(x))) = f'(u(x))\,d(u(x))$, so

$$\int f'(u(x))\,d(u(x)) = \int d(f(u(x))) = f(u(x)) + c.$$

Thus both equal $f(u(x)) + c$. $\square$

Examples 5.2. (i) The example we gave earlier, $\int \sin x \cos x \, dx$. We have

$$\int \sin x \cos x \, dx = \int \sin x \, d(\sin x) = \int_{u \leftarrow \sin x} u \, du = \frac{\sin^2 x}{2} + c.$$

(ii) Another example, $\int e^{2+\cos 2x} \sin 2x \, dx$.

$$\begin{aligned}\int e^{2+\cos 2x} \sin 2x \, dx &= \int e^{2+\cos 2x} \, d(-\tfrac{1}{2}\cos 2x) \\ &= -\frac{1}{2}\int e^{2+\cos 2x} \, d(\cos 2x) \\ &= -\frac{1}{2}\int e^{2+\cos 2x} \, d(2+\cos 2x) \\ &= -\frac{1}{2}\int_{u \leftarrow 2+\cos 2x} e^u \, du = -\tfrac{1}{2}e^{2+\cos 2x} + c.\end{aligned}$$

(iii) We can forgo the involvement of u and integrate directly. For instance:

$$\begin{aligned}\int x\sqrt{2x^2+1}\, dx &= \int \sqrt{2x^2+1}\, d(\tfrac{1}{2}x^2) \\ &= \frac{1}{4}\int \sqrt{2x^2+1}\, d(2x^2) \\ &= \frac{1}{4}\int \sqrt{2x^2+1}\, d(2x^2+1) \\ &= \frac{1}{4}\frac{(2x^2+1)^{3/2}}{3/2} = \tfrac{1}{6}(2x^2+1)^{3/2} + c.\end{aligned}$$

(iv) A few more examples:

$$\int \frac{\cos x}{\sqrt{4-\sin^2 x}}\, dx = \int \frac{d(\sin x)}{\sqrt{4-\sin^2 x}} = \sin^{-1}\left(\frac{\sin x}{2}\right) + c$$

$$\int \frac{\log x}{x}\, dx = \int \log x \, d(\log x) = \frac{\log^2 x}{2} + c$$

$$\int \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{2}\int \frac{d(2x)}{\sqrt{3^2-(2x)^2}} = \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + c$$

(v) Sometimes substitution requires little tricks to make the expression in the differential appear everywhere. For instance:

$$\begin{aligned}\int x\sqrt{2x-1}\, dx &= \frac{1}{4}\int 2x\sqrt{2x-1}\, d(2x) \\ &= \frac{1}{4}\int (2x-1+1)\sqrt{2x-1}\, d(2x-1) \\ &= \frac{1}{4}\int (2x-1)\sqrt{2x-1}\, dx + \frac{1}{4}\int \sqrt{2x-1}\, d(2x-1) \\ &= \frac{1}{4}\frac{(2x-1)^{5/2}}{5/2} + \frac{1}{4}\frac{(2x-1)^{3/2}}{3/2} \\ &= \frac{1}{15}(2x-1)^{3/2}(3x+1) + c.\end{aligned}$$

Notice we rewrote $2x$ as $2x-1+1$ to ensure the appearance of $2x-1$.

Another interesting trick is the following:

$$\begin{aligned}\int \frac{x^5}{\sqrt{x^3+1}}\,dx &= \int \frac{x^3\cdot x^2}{\sqrt{x^3+1}}\,dx \\ &= \frac{1}{3}\int \frac{x^3}{\sqrt{x^3+1}}\,d(x^3) \\ &= \frac{1}{3}\int \frac{x^3+1-1}{\sqrt{x^3+1}}\,d(x^3+1) \\ &= \frac{1}{3}\int (x^3+1)^{1/2}\,d(x^3+1) - \frac{1}{3}\int (x^3+1)^{-1/2}\,d(x^3+1) \\ &= \frac{1}{3}\frac{(x^3+1)^{3/2}}{3/2} - \frac{1}{3}\frac{(x^3+1)^{1/2}}{1/2} \\ &= \frac{2}{9}\sqrt{x^3+1}(x^3-2)+c.\end{aligned}$$

The substitution technique also easily gives us a proof for the two rules from the last section. Indeed, the logarithmic derivative rule is obvious:

$$\int \frac{f'(x)}{f(x)}\,dx = \int \frac{d(f(x))}{f(x)} = \log f(x) + c,$$

and the affine argument rule is also straightforward to obtain:

$$\begin{aligned}\int f(ax+b)\,dx &= \frac{1}{a}\int f(ax+b)\,d(ax) \\ &= \frac{1}{a}\int f(ax+b)\,d(ax+b) \\ &= \frac{1}{a}F(ax+b)+c.\end{aligned}$$

6. Integration by Parts

The second most important formula for integration is the so-called integration by parts formula. Given an integral of the form $\int f(x)\,d(g(x))$, it allows us to swap the roles of f and g, so that we obtain $\int g(x)\,d(f(x))$. This essentially comes from integrating the product rule for differentials.

Indeed, recall that

$$d(u(x)\,v(x)) = u(x)\,d(v(x)) + v(x)\,d(u(x))$$

or more concisely,

$$d(uv) = u\,dv + v\,du.$$

Integrating gives us

$$uv = \int u\,dv + \int v\,du,$$

and the form which we will find useful is

$$\boxed{\int u\,dv = uv - \int v\,du.}$$

Remark 6.1. Integration by parts is useful for integrating products of functions (it comes from the product rule, after all). The difficulty comes in choosing which function should be the u, and which one we should take as part of differential dv.

A good rule of thumb is to pick u to be the first function one comes across according to "ILATE":

I: Inverse trigonometric functions
L: Logarithmic functions
A: Algebraic functions (i.e., x^n)
T: Trigonometric functions
E: Exponential functions

Examples 6.2. (i) In this integral we take $u = x$ (because of the A in IL**A**TE) and dv to be the rest, i.e., $e^{-2x}\,dx$.

$$\begin{aligned}\int xe^{-2x}\,dx &= -\frac{1}{2}\int x\,d(e^{-2x}) \\ &= -\frac{1}{2}\Big(xe^{-2x} - \int e^{-2x}\,dx\Big) \\ &= -\frac{1}{2}\Big(xe^{-2x} + \frac{e^{-2x}}{2}\Big) \\ &= -\frac{e^{-2x}}{4}(2x+1) + c.\end{aligned}$$

(ii) The next example requires two iterations of the formula. Again we pick u to be the algebraic expression, $u = x^2$.

$$\begin{aligned}\int x^2 \sin 2x\,dx &= -\frac{1}{2}\int x^2\,d(\cos 2x) \\ &= -\frac{1}{2}\Big(x^2\cos 2x - \int \cos 2x\,d(x^2)\Big) \\ &= -\frac{1}{2}\Big(x^2\cos 2x - 2\int x\cos 2x\,dx\Big) \\ &= -\frac{1}{2}\Big(x^2\cos 2x - \int x\,d(\sin 2x)\Big) \\ &= -\frac{1}{2}\Big(x^2\cos 2x - \Big(x\sin 2x - \int \sin 2x\,dx\Big)\Big) \\ &= -\frac{1}{2}\Big(x^2\cos 2x - x\sin 2x - \frac{\cos 2x}{2}\Big) \\ &= \frac{1}{4}(\cos 2x + 2x\sin 2x - 2x^2\cos 2x) + c.\end{aligned}$$

(iii) It seems to be the case that something like $\int x^n f(x)\,dx$ requires n iterations of the formula. With this in mind, let us try:

$$\begin{aligned}\int x^{99}\log x\,dx &= \frac{1}{100}\int \log x\,d(x^{100}) \\ &= \frac{1}{100}\Big(x^{100}\log x - \int x^{100}\,d(\log x)\Big) \\ &= \frac{1}{100}\Big(x^{100}\log x - \int x^{99}\,dx\Big) \\ &= \tfrac{1}{10\,000}\,x^{100}(100\log x - 1) + c.\end{aligned}$$

This time we took $u = \log x$ rather than x^{99} in accordance with ILATE, and that's why we didn't need to perform 99 iterations!

(iv) We can use integration by parts to obtain integrals of logarithms and inverse trigonometric functions by treating dx as $d(x)$. For instance,

$$\begin{aligned}\int \log x\,dx &= x\log x - \int x\,d(\log x)\\ &= x\log x - \int dx\\ &= x\log x - x + c.\end{aligned}$$

(v) Another one for good measure:

$$\begin{aligned}\int \tan^{-1} x\,dx &= x\tan^{-1} x - \int x\,d(\tan^{-1} x)\\ &= x\tan^{-1} x - \int \frac{x}{x^2+1}\,dx\\ &= x\tan^{-1} x - \frac{1}{2}\int \frac{d(x^2+1)}{x^2+1}\\ &= x\tan^{-1} x - \frac{1}{2}\log(x^2+1) + c.\end{aligned}$$

(vi) A final popular example, where the same integral shows up on the right-hand side again.

$$\begin{aligned}\int e^x \sin x\,dx &= \int \sin x\,d(e^x)\\ &= e^x \sin x - \int e^x\,d(\sin x)\\ &= e^x \sin x - \int e^x \cos x\,dx\\ &= e^x \sin x - \int \cos x\,d(e^x)\\ &= e^x \sin x - \Big(e^x \cos x - \int e^x\,d(\cos x)\Big)\\ &= e^x \sin x - e^x \cos x - \int e^x \sin x\,dx\end{aligned}$$

At this point, it might feel like we're stuck since we obtained the initial integral again. But just like an equation like $I = 2 - I$ isn't "stuck" (we can easily solve this by taking I to the other side and getting $2I = 2 \Rightarrow I = 1$), we can apply the exact same approach here, and we get that

$$\begin{aligned}2\int e^x \sin x\,dx &= e^x(\sin x - \cos x)\\ \implies \int e^x \sin x\,dx &= \frac{e^x}{2}(\sin x - \cos x) + c.\end{aligned}$$

A final example to nicely conclude these notes is the famous integral $\int \frac{dx}{(1+x^2)^2}$. Noting that

$$d\Big(\frac{1}{f(x)}\Big) = -\frac{1}{2}\,\frac{f'(x)}{f(x)^2}\,dx,$$

we have

$$\begin{aligned}\int \frac{dx}{(1+x^2)^2} &= \int \frac{1+x^2-x^2}{(1+x^2)^2}\,dx \\ &= \int \frac{dx}{1+x^2} - \int \frac{x^2}{(1+x^2)^2}\,dx \\ &= \tan^{-1} x - \frac{1}{2}\int \frac{x}{(1+x^2)^2}\,d(x^2+1) \\ &= \tan^{-1} x + \frac{1}{2}\int x\,d\Big(\frac{1}{1+x^2}\Big) \\ &= \tan^{-1} x + \frac{1}{2}\cdot\frac{x}{1+x^2} - \frac{1}{2}\int \frac{dx}{1+x^2} \\ &= \frac{1}{2}\Big(\tan^{-1} x + \frac{x}{1+x^2}\Big) + c.\end{aligned}$$